TUTTE POLYNOMIALS OF WHEEL-SUNFLOWER GRAPHS AND THEIR APPLICATIONS

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TUTTE POLYNOMIALS OF WHEEL-SUNFLOWER GRAPHS AND THEIR APPLICATIONS

by

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Declaration by the student

I declare that this is my own work and reference has been made to wherever other people's work has been used. I also declare that this work has not been presented or submitted for any award elsewhere.

..... Date: 28-11-05

Mphatso Steve Kamndaya.

CERTIFICATION OF APPROVAL

We declare that this thesis is the result of the candidate's own work, and that where assistance has been sought, this has been acknowlegded and that this thesis has been submitted with our approval.

..... Date: 30-11-05

for Dr. E.G.M. Banda (Supervisor).

..... Date: 30-11-05

Dr. J.J. Namangale (Program Coordinator).

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Dedication

To my wife Kettie.

Abstract

We define a graph and call it a wheel-sunflower graph considered as one of the compartibility graphs that may arise in computer-based network information systems.

Then we study characterizations that permit us to compute Tutte polynomials of wheel-sunflowers using Tutte polynomials of generalized parallel connections. As one of the applications of Tutte polynomials, we characterize wheel-sunflowers by numerical invariants and deduce their T-uniqueness - that is, graphs determined up to isomorphism by their Tutte polynomials.

We also apply the theory of Tutte polynomials to determine the component numbers of links corresponding to wheel-sunflower graphs and show how these numbers change by removing certain edges.

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CHAPTER 1

Introduction

Some of the research problems in matroid theory involve finding formulas that reduce computations of the Tutte polynomials of matroids to simpler computations [**3**] and searching matroids that are determined by their Tutte polynomials [**11, 12, 13**]. Some other research problems involve applying the theory of Tutte polynomials in other fields. For example, in knot theory, Tutte polynomials are used to classify knots and links associated with planar graphs [**15**].

This thesis focuses precisely on the issues above. First we are going to introduce a new class of graphs called wheel-sunflower graphs and then compute their Tutte polynomials. Such type of graphs may arise in network information systems and parameters encoded in their Tutte polynomials may be used to solve some of the many problems that arise in them. One such problem is that of network reliability [**8**]. The second issue is that we are going to establish that wheel-sunflower graphs can be completely determined from their Tutte polynomials. Graphs with such a property are called T-unique. T-uniqueness is one of the applications of Tutte polynomials. Lastly we are going to apply the theory of Tutte polynomials to classify links whose associated graphs are wheel-sunflowers and variations of wheel-sunflowers.

1.1. Overview of thesis

The structure of this thesis is as follows. In Chapter 2, we introduce and define a wheel-sunflower graph. Then we give characterizations that enable us to use Tutte polynomials of generalized parallel connections to compute Tutte polynomials of wheel-sunflowers. In Chapter 3, we give several parameters of wheel-sunflowers like the number of cycles of lengths $3, 4$ and 5, that can be determined from the Tutte polynomial. These parameters are used to prove T-uniqueness of wheel-sunflowers. Chapter 4 focuses on one of the applications of Tutte polynomials. It gives component numbers of links whose associated

graphs are wheel-sunflowers. Furthermore it gives results on how the component numbers of links change on the removal of certain edges in a wheel-sunflower graph.

In some sections of this thesis we give new results unless otherwise stated. Namely, the definition of a wheel-sunflower in Section 2.1; properties of wheel-sunflowers in Sections 2.1.1 and 3.2; T-uniqueness of wheel-sunflowers in Sections 3.3 and 3.2; component numbers of links associated with wheel-sunflowers in Section 4.2; and component number of links corresponding to a graph called a fan-sunflower graph in Section 4.2.1.

Most of the work in Chapter 2 is not new. For instance in Section 2.1.2 we review the definitions of modular flats and Möbius functions [**3, 6**] and we apply these definitions to wheel-sunflowers. In Section 2.2 we review different formulas of the characteristic polynomial, Tutte polynomial and the weighted characteristic polynomial as given by Bonin and de Mier [**3**]. The relationships of these formulas are used to work out the Tutte polynomials of generalized parallel connections.

The work in Sections 2.3.2 and 2.3.3 is entirely found in [**3**]. We have only shown the characterizations that will enable us to use the same on the cycle matroid of an n-wheel-sunflower.

Most of the work in Chapter 3 corresponds to routine proofs of results on T-uniqueness of graphs found in [**12**]. In the last chapter, Chapter 4, Section 4.1 reviews constructions of links from planar graphs. Then we apply this to wheel-sunflower graphs.

1.2. Basic definitions

This section is devoted to reviewing the basic terminology, notation and operations of graph theory, matroid theory and knot theory that have been used in this thesis.

1.2.1. Graph theory.

We review the notation used and definitions that appear frequently in this thesis. For all undefined notations we refer to any introductory book such as [**20**].

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$ and a relation that associates with each edge two vertices, not necessarily distinct, called its *endpoints*. If there is no confusion, we write $V(G)$ and $E(G)$ as V and E respectively. An edge of G with endpoints v and w is denoted by $\{v, w\}$. In this thesis we study graphs without

isolated vertices, but possibly with loops. An isolated vertex is a vertex of degree 0 and a loop is an edge whose endpoints are equal.

A graph G is *connected* if each pair of vertices in G belongs to a *path*; otherwise G is *disconnected*. A path from a vertex v to a vertex u is a sequence of distinct alternating vertices and edges, $v_1, e_1, v_2, e_2, \cdots, e_{k-1}, v_k$, such that $v_1 = v$, $v_k = u$ and for all $i = 1, 2, \cdots, k-1$, e_i is incident with v_i and v_{i+1}. The *connectivity* $\kappa(G)$ of connected graph G is the smallest number of vertices whose removal disconnects G. When $\kappa(G) \geq 2$, the graph is said to be 2-*connected*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. An *induced subgraph* is a subgraph obtained by deleting a set of vertices. If G is a graph and $A \subset V(G)$ we denote by $G|A$ the subgraph of G *induced* by A.

For every subset $A \subseteq E$, its *rank* is $r(A) = n - k(G|A)$, where $n = |V(A)|$ and $k(G|A)$ is the number of connected components of $G|A$. We write $r(G)$ instead of $r(E)$. The *nullity* of an edge-set is $n(A) = |A| - r(A)$.

A cycle of length n is called an $n-$cycle and denoted by C_n; for $n = 3, 4, 5$ we also refer to them as *triangles*, *squares* and *pentagons* respectively. The minimum n such that G has an $n-$cycle is called the *girth* of G denoted by $g(G)$. A *chord* of a cycle C is an edge joining two non consecutive vertices of C. A graph with no cycle is *acyclic*. A *forest* is an acyclic graph.

A graph G is *planar* if it can be drawn in the plane in such a away that no two edges meet each other except at a vertex to which they are both incident. Any such drawing is called a *plane drawing* of G. Wheel-sunflower graphs to be studied in this thesis are planar.

1.2.2. Matroid theory.

In this section we review the basic definitions and notation used in this thesis for matroids. We refer to Oxley's book [**17**] for further details and proofs of the results in this section. Matroids can be defined in several ways. In this thesis we define them in terms of independent sets.

A *matroid* is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E satisfying the following properties:

($I1$) $\emptyset \in \mathcal{I}$.

($I2$) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

($I3$) If $I, I' \in \mathcal{I}$ and $|I'| < |I|$, then there exists an element $x \in I - I'$ such that $I' \cup x \in \mathcal{I}$.

The set E is called the *ground set* of M, and the sets in $\mathcal{I}$ are the *independent sets.* We give an example of a matroid.

EXAMPLE 1.2.1. Let $G = (V, E)$ be a graph and let $\mathcal{A}$ be the set of acyclic subsets of E. Then $(E, \mathcal{A})$ is a matroid denoted by $M(G)$. We call $M(G)$ the *cycle matroid* of G.

An *isomorphism* between matroids M and N is a *bijection* $\varphi : E(M) \longrightarrow E(N)$ such that for each subset $I \subseteq E(M)$, I is independent in M if and only if $\varphi(I)$ is independent in N. The subsets of the ground set that are not independent are called *dependent;* minimal dependent sets are *circuits* and maximal independent sets are *bases.* For a graph G, the *girth* of a cycle matroid $M = M(G)$ is the number of elements of its smallest circuit, and it is denoted by $g(M)$. An element x such that $\{x\}$ is a circuit is called a *loop*; hence loops are in no basis. If an element belongs to every basis then it is called a *coloop.* If a set contains a basis, it is called *spanning.*

For any set $X \subseteq E$, the *rank* of X is the size of the largest independent set in X, and is denoted by $r(X)$ and the *nullity* of X is $n(X) = |X| - r(X)$. If $r(\{x\}) = r(\{y\}) = r(\{x, y\}) = 1$, we say that the elements x and y are *parallel.*

A matroid *invariant* is a function f defined on matroids M and N such that $f(M) = f(N)$ whenever M and N are isomorphic. For instance, the rank, the number of bases, and the girth are matroid invariants.

Given two matroids M_1 and M_2 with disjoint ground sets, E_1 and E_2, the matroid $M_1 \oplus M_2$ has as ground set $E_1 \cup E_2$ and as independent sets $\mathcal{I}(M_1 \oplus M_2) = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}$. Such a matroid $M_1 \oplus M_2$ is called the *direct sum* of M_1 and M_2. A set $X \subseteq E$ is called a *flat* if $r(X \cup e) = r(X) + 1$ for all $e \notin X$. The *closure* of a set $X \subseteq E$ is the smallest flat containing X. Furthermore, if there is a set of flats $\{X_0, X_1, \cdots, X_r\}$ of a matroid M such that, for $i \in \{0, 1, 2, \cdots, r\}, r(X_i) = i$ and for $i \in \{1, 2, \cdots, r\}, X_{(i-1)} \subseteq X_i$, then $\{X_0, X_1, \cdots, X_r\}$ is called a *saturated chain of flats* of M. Given two matroids M_1 and M_2 with a common flat N on the ground sets, E_1 and E_2 respectively, the matroid $P_N(M_1, M_2)$ on the ground set $E_1 \cup E_2$ is called the *generalized parallel connection* of M_1 and M_2.

Definitions 1.2.2 and 1.2.3 are used to compute Tutte polynomials of matroids using generalized parallel connections.

DEFINITION 1.2.2. A *relation* " $\leq$ " is a partial order on a set X if it has:

(i) reflexivity ($x \leq x$ for all $x \in X$),
(ii) antisymmetry ($x \leq y$ and $y \leq x$ implies that $x = y$),
(iii) transitivity ($x \leq y$ and $y \leq z$ implies that $x \leq z$).

DEFINITION 1.2.3. Boolean algebra B_n is the partial order on subsets defined by inclusion $\subseteq$ - that is, the Boolean algebra B_n, has elements as all subsets of a set $\{1, 2, \cdots, n\}$ and $\subseteq$ as an order relation.

Sets with a partial order and an order relation are called partially ordered sets (posets). So the Boolean algebra is an example of a poset.

1.2.3. The Tutte polynomial.

In this section we give the definition of Tutte polynomial and outline some of the basic operations that are used to compute the Tutte polynomial of a matroid. We also include the definitions of T-equivalent and T-uniqueness. The former is used to define graphs with same Tutte polynomial while the later is used to define a graph G such that for all graphs H, that are T-equivalent to G are also isomorphic to G. For a thorough introduction we refer to [**8, 10, 12, 13**].

Much as the Tutte polynomial is defined in several ways, we are going to use the following definition in this thesis. The *Tutte polynomial* of a matroid M with ground set E is defined as: $t(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}$.

The Tutte polynomial of a matroid is naturally extended to a graph G as $t(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$. Some of the basic operations that are used to compute the Tutte polynomial are deletion and contraction. The *deletion* $M \backslash e$ is the matroid on $E - e$ having as independent sets $\mathcal{I}(M \backslash e) = \{I : I \in \mathcal{I}(M), e \notin I\}$. The *contraction* of an edge e with end points u and v denoted by $M(G/e)$ from the cycle matroid of G, is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u and v and the resulting cycle matroid $M(G/e)$ has one edge less than $M(G)$. The *Tutte polynomial* of a cycle matroid $M = M(G)$ can be computed by applying the following operations recursively:

(i) If $E = \emptyset$, then $t(M; x, y) = 1$;

(ii) If e is a coloop, then $t(M; x, y) = x\ t(M/e; x, y)$;

(iii) If e is a loop, then $t(M; x, y) = y\ t(M\backslash e; x, y)$;

(iv) If e is neither a coloop nor a loop, then $t(M; x, y) = t(M\backslash e; x, y) + t(M/e; x, y)$.

We say that two graphs G_1 and G_2 are *T-equivalent* if they have the same Tutte polynomial. Furthermore, a graph G is *T-unique* if any other graph having the same Tutte polynomial as G is necessarily isomorphic to G. Several well known families of graphs such as wheels, squares of cycles, complete multipartite graphs, ladders, Möbius ladders and hypercubes are known to be T-unique [**11**].

1.2.4. Knot theory.

In this section we summarize the knot theory used in this thesis. We give the basic definitions, notations and some of the operations on knots and links. We refer to Adam's book [**1**] and Murasugi's paper [**16**] for a thorough introduction to the subject.

A *knot* is a simple closed polygon in three dimensional space and it has a finite number of vertices and edges. In the plane a knot is shown as a two dimensional figure with self crossings known as the *knot diagram.* The knot diagram which does not give any information about the type of crossing points and orientation is called a *knot universe.* A knot diagram which is composed of one component is referred to as a *knot* and if it is composed of more than one component it is called a *link diagram.*

Link diagrams can be of two types: oriented and unoriented. An oriented link diagram consists of directed lines while an unoriented link diagram has no directed lines. We refer the reader to [**1**] for a nice introduction to the subject. In this thesis we have considered the case of unoriented link diagrams which correspond to wheel-sunflower graphs and their variations. We are going to call the lines of unoriented link diagrams as components of a link. For such an unoriented link diagram $K(G)$, that corresponds to a planar graph G, its linking number $L(K(G))$ is determined by applying the theory of Tutte polynomials.

CHAPTER 2

Tutte polynomials of wheel-sunflowers

In this chapter we are going to study the characterizations of wheel-sunflowers that will enable us to compute the Tutte polynomials of wheel-sunflowers using Tutte polynomials of generalized parallel connections found in [**3**].

2.1. The wheel-sunflower graph

This section introduces wheel-sunflower graphs and gives the formal definition of wheel-sunflower graphs. Then we study some of the properties of wheel-sunflowers with regard to their edges, modular flats and computation of the möbius function.

The *join* of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{\{u, v\} : u \in V(G), v \in V(H)\}$. For instance, an n-wheel graph W_n, is the join of a cycle graph C_n and a complete graph K_1 placed at the "middle" of C_n. We write $W_n = K_1 \vee C_n$. Edges from C_n are called *rim* edges of W_n and the edges added are called *spokes*. A middle vertex is called a *hub* [**15**].

The n-wheel-sunflower graph denoted by WS_n is built on an n-wheel graph, for $n \geq 2$, by adding n "outer" triangles according to the following rule: One edge from each of the "outer" triangles together with its endpoints are identified from the rim edges of an n-wheel W_n. Definition 2.1.1 gives the formal definition of an n-wheel-sunflower.

DEFINITION 2.1.1. Let F be a group of integers modulo n and $W_n = C_n \vee K_1$. Then for $n \geq 2$, an n-wheel-sunflower denoted by WS_n is a graph with:

$$V(WS_n) = V(C_n \vee K_1) \cup \{b_i : i \in F;\ b_i \notin W_n\};$$

$$E(WS_n) = E(C_n \vee K_1) \cup \{\{b_i, a_i\}, \{b_i, a_{i+1}\} : i \in F;\ a_i \in V(C_n);\ b_i \notin W_n\}.$$

An edge set of the form $\{\{a_i, b_i\}, \{b_i, a_{i+1}\} : i \in F,\ a_i \in V(C_n),\ b_i \notin W_n\}$ is called a *petal*, and any "outer" vertex $b_i \notin W_n$ is called an *apex* of an n-wheel-sunflower. Essentially then, the number of petal edges added to an n-wheel-sunflower is $2n$ since each of the n

vertices from the rim is joined to two additional edges with a common vertex. Figure 2.1 is an example of a 6-wheel-sunflower.

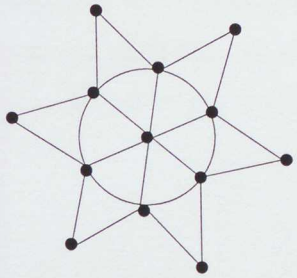

FIGURE 2.1. A 6-wheel-sunflower graph

A wheel-sunflower graph can be regarded as one of the *compartibility* graph. A *compartibility* graph is a graph in which vertices represent objects being arranged, and edges correspond to those pairs of objects which are compartible in some way [**21**]. For instance wheel-sunflower graphs can arise in computer-based network information systems in which vertices from the rim correspond to computers that communicate in a ring and directly to one large central server computer that coresponds to the hub. The apex can represent smaller server computers for back-up information from an adjacent pair of computers corresponding to adjacent pairs of vertices from the rim. All edges in these cases represent communication lines.

2.1.1. Edges of an n-wheel-sunflower.

In this section we are going to study some properties that are immediate from the vertex set $V(WS_n)$ and edge set $E(WS_n)$ of wheel-sunflower graphs. We start by presenting Lemma 2.1.2 which will be use to prove Proposition 2.1.3 on the number of edges in an n-wheel-sunflower.

LEMMA 2.1.2. *Let WS_n be an n-wheel-sunflower. Then $|E(WS_{(n+1)})| = |E(WS_n)| + 4$.*

PROOF. From Definition 2.1.1, to construct an $(n+1)$-wheel-sunflower $WS_{(n+1)}$, from WS_n we add the following four edges to $E(WS_n)$: one rim edge, one spoke and two petal edges. □

PROPOSITION 2.1.3. *Let WS_n be an n-wheel-sunflower. Then $|E(WS_n)| = 4n$.*

PROOF. By using induction on n. For $n = 2$, we get WS_2 which has 4 petal edges, 2 rim edges which are parallel and 2 spokes, making a total of 8 edges. Assume that the proposition holds for $n = k$ where $n > 2$. Then $|E(WS_k)| = 4k$. Consider the case $n = k + 1$. Then by Lemma 2.1.2, we get $|E(WS_{(k+1)})| = |E(WS_k)| + 4$. By the induction hypothesis we have: $|E(WS_{(k+1)})| = 4k + 4 = 4(k + 1)$. Hence the proposition holds for any n. □

2.1.2. Modular flats.

In this section we summarize key results about modular flats which are used in Proposition 2.1.7. We start by reviewing the following definitions found in [**17**].

Let $M = M(G)$ where G is a graph. If G is connected, then the rank $r(M)$ of a matroid M is given by $r(M) = |V(G)| - 1$. This can be extended as follows. If G has $\omega(G)$ connected components, then $r(M) = |V(G)| - \omega(G)$. It follows that if $X \subseteq E(G)$, then $r(X) = |V(G[X])| - \omega(G[X])$.

EXAMPLE 2.1.4. From the definition of an n-wheel-sunflower WS_n, in Section 2.1 we have $r(M(WS_n)) = |V(WS_n)| - 1 = 2n$.

Recall, from Section 1.2, that a set $X \subseteq E$ is a flat if $r(X \cup e) = r(X) + 1$ for all $e \notin X$. Flats of ranks $1, 2, 3$, and $r(M) - 1$ are called points, lines, planes and hyperplanes respectively. The following definition of a modular flat in a matroid and its justification are found in [**6, 9**].

DEFINITION 2.1.5. A flat A of a matroid M is modular if for each flat X of M, we have $r(A) + r(X) = r(A \cup X) + r(A \cap X)$.

In this thesis we are going to use Theorem 2.1.6 found in [**6**], to prove Proposition 2.1.7.

THEOREM 2.1.6. *Let M be a matroid with its rank as $r(M)$ and the rank of the flat A of M is k. Then A is modular if and only if A intersects every flat of rank $r(M) - k + 1$ in a flat of positive rank. In particular, a hyperplane H of M is modular if and only if H intersects every line of M in a flat of positive rank.*

PROPOSITION 2.1.7. *Let W_n be an n-wheel in an n-wheel-sunflower WS_n. Then $M(W_n)$ is a rank-n modular flat in $M(WS_n)$.*

PROOF. From the definition of an n-wheel-sunflower WS_n, we have: $r(M(WS_n)) = |V(WS_n)| - 1 = 2n$. Hence $M(WS_n)$ is a rank-$2n$ matroid. Furthermore, $r(M(W_n)) = |V(W_n)| - 1 = n$. To use Theorem 2.1.6 here we let $k = n$. Then we need to show that $M(W_n)$ intersects every flat of rank $2n-n+1$ in a flat of positive rank; that is rank-$(n+1)$ flats. So let X be a flat of rank $n+1$ in $M(WS_n)$. All rank-$(n+1)$ flats in $M(WS_n)$ have at least one edge from $E(WS_n)$ which is either a spoke or a rim edge. Spokes and rim edges are edges of an n-wheel W_n. Therefore the intersection $W_n \cap X$ contains at least a spoke or a rim edge. Spokes and rim edges are rank-1 flats in G_n. Hence $r(W_n \cap X) \geq 1$ as required. □

2.1.3. Möbius function.

In this section we give a brief guide, to the reader, on one of the invariants that can be obtained from the Tutte polynomial called the Möbius function. We summarize ideas on how the Möbius function denoted by μ is defined on a set. For a thorough review and basic properties of Möbius functions, we refer to [**6, 23**].

Recall Definition 1.2.3 and the meaning of a poset from Section 1.2.2 which we use now.

DEFINITION 2.1.8. Let P be a poset whose intervals are finite and $Int(P)$ be set of its intervals. The Möbius function, $\mu : Int(P) \longrightarrow \mathbf{Z}$, is defined recursively by

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x,z), & \text{if } x < y. \end{cases}$$

In Example 2.1.9 we explain how to compute $\mu(\emptyset, F)$, where F is a flat of a matroid M. We will need this to compute the Tutte polynomial of the cycle matroid $M(WS_n)$ of an n-wheel-sunflower using generalized parallel connection.

EXAMPLE 2.1.9. A rank-n wheel W_n consists of edges which can be ordered in the following manner $\{1, 2, 3, \cdots, 2n\}$; see Figure 2.2. And then by defining the Boolean algebra, B_{2n}, of the set $\{1, 2, \cdots, 2n\}$ and $\subseteq$ as an order relation, we can evaluate $\mu(\emptyset, F)$ where $F \in B_{2n}$. It is already known that $\mu(\emptyset, F) = (-1)^{|F|}$, where $|F|$ is the cardinality of F [**3, 23**]. For example, from Figure 2.2:

(i) All the six triangles are rank-2 flats of cardinality 3. Therefore their corresponding Möbius function is $(-1)^3 = -1$;

(ii) All the six cycles of length five with two chords with a common vertex are rank-4 flats of cardinality 7. Therefore their corresponding Möbius function is $(-1)^7 = -1$;

(iii) A 6-wheel itself is a rank-6 flat of cardinality 12. Therefore its corresponding Möbius function is $(-1)^{12} = 1$.

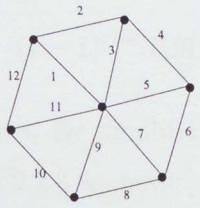

FIGURE 2.2. A 6-wheel

2.2. Tutte and weighted characteristic polynomials

In this section we review formulations of the Tutte polynomial and the characteristic polynomial as found in [3]. Recall the definition of the Tutte polynomial from Section 1.2.3. The Tutte polynomial $t(M; x, y)$ of a matroid M on the set E is given by:

$$t(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}. \tag{2.2.1}$$

The characteristic polynomial $\chi(M; \lambda)$ of M is, up to sign, a special evaluation of the Tutte polynomial of M : $\chi(M; \lambda) = (-1)^{r(M)} t(M; 1-\lambda, 0)$. The characteristic polynomial can be formulated in a variety of ways. Equation 2.2.1 yields the following Boolean expression of the characteristic polynomial. $\chi(M, \lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(M)-r(A)}$. This is called the Boolean expression, because $(-1)^{|A|}$ is an evaluation of Boolean algebra, B_n, using $\mu(A)$ where $A \in B_n$. Hence the characteristic polynomial can also be expressed in the following way: $\chi(M; \lambda) = \sum_{\text{flats } F \text{ of } M} \mu(\emptyset, F) \lambda^{r(M)-r(F)}$, where μ is the Möbius function of M [23]. It follows from any of these formulas that $\chi(M; \lambda) = 0$ if M has loops. This is because one of the properties of deletion-contraction recursion states that if e is a loop in a matroid M, then $t(M; x, y) = y\ t(M \backslash e; x, y)$. And recall that $y = 0$ in the characteristic polynomial. Thus according to Brylawski, for a contraction M/Z of M, the characteristic polynomial $\chi(M/Z; \lambda)$ is nonzero only if Z is a flat of M [3]. Definition 2.2.1 is due to Brylawski [3].

DEFINITION 2.2.1. The weighted characteristic polynomial, $\bar{\chi}(M; x, y)$ of M in terms of the characteristic polynomial of contractions of M by flats is given by $\bar{\chi}(M; x, y) = \sum_{\text{flats F of M}} x^{|F|} \chi(M/F; y)$.

The following well known formulas make precise the statement that the Tutte polynomial $t(M;x,y)$ and the weighted characteristic polynomial $\bar{\chi}(M;x,y)$ are equivalent [3]:

$$t(M;x,y) = \frac{\bar{\chi}(M;y,(x-1)(y-1))}{(y-1)^{r(M)}}; \tag{2.2.2}$$

$$\bar{\chi}(M;x,y) = (x-1)^{r(M)}t(M;\frac{y}{x-1}+1,x). \tag{2.2.3}$$

We illustrate this equivalence as follows. By simplifying the numerator $\bar{\chi}(M;y,(x-1)(y-1))$ of Equation 2.2.2 using Equation 2.2.3 we have:

$$\begin{aligned}\bar{\chi}(M;y,(x-1)(y-1)) &= (y-1)^{r(M)}t(M;\frac{(x-1)(y-1)}{y-1}+1,y)\\ &= (y-1)^{r(M)}t(M;x,y).\end{aligned}$$

By substituting this result $(y-1)^{r(M)}t(M;x,y)$ in Equation 2.2.2 we get $t(M;x,y)$ as required.

2.3. Generalized parallel connections

This section defines the cycle matroid of an n-wheel-sunflower WS_n using generalized parallel connection. We are going to use the following assumptions for the matroids M_1 and M_2 on the ground sets E_1, and E_2 respectively, found in [3]:

(G1) $M_1|T = M_2|T$, where $T = E_1 \cap E_2$;
(G2) $cl_{M_1}T$ is a modular flat of M_1; and
(G3) $cl_{M_2}T$ is a modular flat of M_2.

DEFINITION 2.3.1. Let N denote the common restriction $M_1|T = M_2|T$. The generalized parallel connection of M_1 and M_2 at T is a matroid denoted as $P_N(M_1,M_2)$ whose flats are precisely the subsets A of $E_1 \cup E_2$ such that $A \cap E_1$ is a flat of M_1 and $A \cap E_2$ is a flat of M_2.

Equivalently, the flats of $P_N(M_1,M_2)$ are the subsets of $E_1 \cup E_2$ of the form $A_1 \cup A_2$ where A_1 and A_2 are flats of M_1 and M_2, respectively, and $A_1 \cap T = A_2 \cap T$ [3]. We shall use this second formulation of flats in Section 2.3.3. We can now give the definition of the cycle matroid of wheel-sunflowers using generalized parallel connection as Proposition 2.3.2.

Proposition 2.3.2 is required to apply the results of Tutte polynomials of generalized parallel connection directly to the cycle matroid of an n-wheel-sunflower denoted as $M(WS_n)$. We are going to identify the two constituent matroids of $M(WS_n)$ as depicted in Figure 2.3.

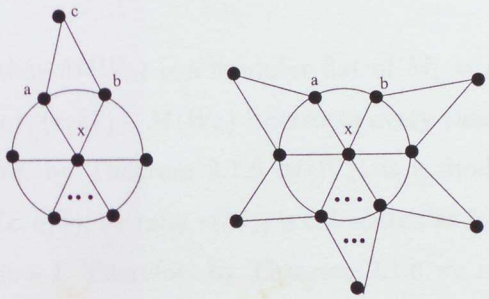

FIGURE 2.3. Constituent matroids of WS_n

PROPOSITION 2.3.2. *Let $WS_n \setminus \{\{a,c\},\{c,b\}\}$ be an n-wheel-sunflower with a missing petal $\{\{a,c\},\{c,b\}\}$, such that $\{a,b\}$ is a rim edge and $W_n \cup \{\{a,c\},\{c,b\}\}$ be an n-wheel with a petal $\{\{a,c\},\{c,b\}\}$, such that $\{a,b\}$ is the same rim edge as the first. Let $M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M(W_n \cup \{\{a,c\},\{c,b\}\})$ be the two matroids on the ground sets $E_1 = E(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $E_2 = E(W_n \cup \{\{a,c\},\{c,b\}\})$ respectively. Then $M(WS_n)$ is the generalized parallel connection $P_N(M_1, M_2)$ of $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$.*

PROOF. The intersection $E(WS_n \setminus \{\{a,c\},\{c,b\}\}) \cap E(W_n \cup \{\{a,c\},\{c,b\}\})$ of E_1 and E_2 is an edge set $E(W_n)$ of an n-wheel. So let $T = M(W_n)$. Then:

$$\begin{aligned}
M_1|T &= M(WS_n \setminus \{\{a,c\},\{c,b\}\})|M(W_n) \\
&= M(WS_n \setminus \{\{a,c\},\{c,b\}\}) \setminus (E_1 - E(W_n)) \\
&= M(WS_n \setminus \{\{a,c\},\{c,b\}\} \setminus (\bigcup_{i=1}^{n} \{\{a_i,b_i\},\{b_i,a_{i+1}\}\} \setminus \{\{a,c\},\{c,b\}\}) \\
&= M(W_n), \\
M_2|T &= M(W_n \cup \{c_1,c_2\})|M(W_n) \\
&= M(W_n \cup \{\{a,c\},\{c,b\}\}) \setminus (E_2 - E(W_n)) \\
&= M(W_n \cup \{\{a,c\},\{c,b\}\} \setminus \{\{a,c\},\{c,b\}\}) \\
&= M(W_n).
\end{aligned}$$

Let e be an edge in $E(WS_n)$. If $e \notin E(W_n)$, then e is one of the petal edges in either $E(WS_n \setminus \{\{a,c\},\{c,b\}\})$ or $E(W_n \cup \{\{a,c\},\{c,b\}\})$. Furthermore $r(W_n \cup e) = r(W_n) + 1$. Hence $M(W_n)$ is a flat of both $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$.

Lastly we need to show that $M(W_n)$ is a modular flat of $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$. $M(W_n)$ intersects every rank-2 flat of $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$. Therefore, by Theorem 2.1.6 $M(W_n)$ is a modular flat of M_2. And for $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$, its rank $r(M_1)$ is calculated as follows: $r(M_1) = |V(WS_n \setminus \{\{a,c\},\{c,b\}\})| - 1 = 2n - 1$. Therefore by Theorem 2.1.6 we need to show that $M(W_n)$ intersects every flat of rank $(2n-1) - n + 1 = n$ in a positive rank, which is $M(W_n)$ itself. So $M(W_n)$ is a modular flat of $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$. □

To apply the results of Tutte polynomials of generalized parallel connection we require that $M(W_n)$ be a *maximal* common restriction of both $M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M(W_n \cup \{\{a,c\},\{c,b\}\})$. This is shown in Proposition 2.3.3.

PROPOSITION 2.3.3. *$M(W_n)$ is the maximal common restriction of $M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M(W_n \cup \{\{a,c\},\{c,b\}\})$.*

PROOF. We use contradiction. Suppose that $M(W_n)$ is not maximal. Then there is an e which is not in $E(W_n)$ such that the intersection $E(WS_n \setminus \{\{a,c\},\{c,b\}\}) \cap E(W_n \cup \{\{a,c\},\{c,b\}\})$ has e as one of its elements. Since e is not in $E(W_n)$ then e must be a petal edge. But from Proposition 2.1.3, the number of petal edges in $E(WS_n \setminus \{\{a,c\},\{c,b\}\})$ is $2n-2$ and number of petal edges in $E(W_n \cup \{\{a,c\},\{c,b\}\})$ is 2. Therefore if e is in the intersection then total number of petal edges in n-wheel-sunflower will be $(2n-2)+2-1$, which simplifies to $2n-1$. This is a contradiction because the total number of petal edges in an n-wheel-sunflower WS_n is $2n$. □

Throughout this chapter, when two matroids M_1 and M_2 are under consideration we are going to denote their common restriction by N. But for the restriction to one matroid M we are going to denote it by $M|T$.

2.3.1. Properties of generalized parallel connections.

Our results on the Tutte polynomials of wheel-sunflowers follow directly from Tutte polynomials of generalized parallel connections. Hence the rest of this section focuses on

properties of generalized parallel connections found in [3]. First we turn to the ranks of flats in $P_N(M_1, M_2)$.

Recall the meaning of the terms; rank, closure, flat and saturated chain of flats from Section 1.2.2. The rank of a flat A of a matroid is perceived as the number of flats other than closure $cl(\emptyset)$ of an empty set in a saturated chain of flats from $cl(\emptyset)$ to A [3]. Using this perspective on rank together with the definition of the flats of $P_N(M_1, M_2)$ it follows that the rank of a flat $A_1 \cup A_2$ of $P_N(M_1, M_2)$, where A_1 and A_2 are flats of M_1 and M_2, respectively, with $W = A_1 \cap T = A_2 \cap T$, is given by Equation 2.3.1:

$$\begin{aligned} r(A_1 \cup A_2) &= r_N(W) + (r_{M_1}(A_1) - r_N(W)) + (r_{M_2}(A_2) - r_N(W)) \\ &= r_{M_1}(A_1) + r_{M_2}(A_2) - r_N(W). \end{aligned} \tag{2.3.1}$$

In particular if $A_1 = M(WS_n \setminus \{\{a, c\}, \{c, b\}\})$ and $A_2 = M(W_n \cup \{\{a, c\}, \{c, b\}\})$ then, the rank of a matroid $P_N(M_1, M_2)$ is given by Equation 2.3.2:

$$r(P_N(M_1, M_2)) = r(A_1 \cup A_2) = r(M_1) + r(M_2) - r(N). \tag{2.3.2}$$

Notice that we can use equation 2.3.2 to calculate the rank of $M(WS_n)$. Example 2.3.4 illustrates this rank calculation on the cycle matroid of an n-wheel-sunflower.

EXAMPLE 2.3.4. Let $M_1 = M(WS_n \setminus \{\{a, c\}, \{c, b\}\})$, where $WS_n \setminus \{\{a, c\}, \{c, b\}\})$ is an n-wheel-sunflower with a missing petal $\{\{a, c\}, \{c, b\}\}$ and $M_2 = M(W_n \cup \{\{a, c\}, \{c, b\}\})$, where $W_n \cup \{\{a, c\}, \{c, b\}\}$ is an n-wheel with a petal $\{\{a, c\}, \{c, b\}\}$. Then by Proposition 2.3.3, the common restriction, N, of both M_1 and M_2 is the cycle matroid of an n-wheel. Hence:

$$\begin{aligned} r(N) &= |V(W_n)| - 1 \\ &= n, \\ r(M_1) &= |V(WS_n \setminus \{\{a, c\}, \{c, b\}\})| - 1 \\ &= 2n - 1, \\ r(M_2) &= |V(W_n \cup \{\{a, c\}, \{c, b\}\})| - 1 \\ &= n + 1. \end{aligned}$$

By substituting these in equation 2.3.2 we get

$$\begin{aligned} r(P_N(M_1, M_2)) &= r(M_1) + r(M_2) - r(N) \\ &= 2n \\ &= r(M(WS_n)). \end{aligned} \tag{2.3.3}$$

Proposition 2.3.5 is found in [**3**] and we will use it in Section 2.3.2.

PROPOSITION 2.3.5. *Assume that M_1, M_2 are matroids on the ground sets E_1, E_2, and that conditions (G1)-(G3) on generalized parallel connections hold. Then the following property holds: $P_N(M_1, M_2)/T = (M_1/T) \oplus (M_2/T)$.*

2.3.2. Characteristic polynomial of $P_N(M_1, M_2)$.

Recall that by Proposition 2.3.2 we can define $M(WS_n)$ as a generalized parallel connection $P_N(M_1, M_2)$ of $M_1 = M(WS_n \setminus \{\{a, c\}, \{c, b\}\})$ and $M_2 = M(W_n \cup \{\{a, c\}, \{c, b\}\})$, such that $N = M_1|T = M_2|T$, where $T = M(W_n)$ is the cycle matroid of an n-wheel, W_n. To apply the definition of the weighted characteristic polynomial to a generalized parallel connection $P_N((M_1, M_2)$ we need to know the characteristic polynomial of contractions of $P_N(M_1, M_2)$ by flats. Lemma 2.3.6 given by Bonin in [**3**], addresses these contractions.

LEMMA 2.3.6. *Assume that M_1, and M_2 are matroids on the ground sets E_1, E_2, and that conditions $(G1) - (G3)$ hold. For a flat F of $P_N(M_1, M_2)$ with $F \cap E_1 = A_1$, $F \cap E_2 = A_2$, and $F \cap T = W$, we have:*

$$P_N(M_1, M_2)/F = P_{N/W}(M_1/A_1, M_2/A_2). \tag{2.3.4}$$

Turning to the characteristic polynomial, Proposition 2.3.5 and basic properties of the characteristic polynomial give by Equation 2.3.5 [**3**]:

$$\begin{aligned} \chi(P_N(M_1, M_2)/T; \lambda) &= \chi(((M_1/T) \oplus (M_2/T)); \lambda) \\ &= \chi(M_1/T; \lambda)\chi(M_2/T; \lambda). \end{aligned} \tag{2.3.5}$$

The following theorem is given by Brylawski [**3**], and it gives the characteristic polynomial of $P_N(M_1, M_2)$.

THEOREM 2.3.7. *Assume that M_1, and M_2 are matroids on the ground sets E_1, E_2, and that conditions (G1)-(G3) hold. If the maximal common restriction N of M_1 and M_2 has no loops, then $\chi(P_N(M_1, M_2); \lambda) = \frac{\chi(M_1;\lambda)\chi(M_2;\lambda)}{\chi(N;\lambda)}$.*

2.3.3. Weighted characteristic polynomial of $P_N(M_1, M_2)$.

Recall that the flats of $M(WS_n) = P_N(M_1, M_2)$ are the subsets of $E(WS_n \setminus \{\{a,c\},\{c,b\}\}) \cup E(W_n \cup \{\{a,c\},\{c,b\}\})$ of the form $A_1 \cup A_2$ where A_1 and A_2 are flats of $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$, respectively, and $A_1 \cap T = A_2 \cap T$. Also, recall that from Definition 2.2.1 we have $\bar{\chi}(M;x,y) = \sum_{\text{flats F of M}} x^{|F|}\chi(M/F;y)$.

Using this definition of the weighted characteristic polynomial $\bar{\chi}(M;x,y)$ and the formulation of the flats of the generalized parallel connection $P_N(M_1, M_2)$, it follows that the weighted characteristic polynomial $\bar{\chi}(P_N(M_1, M_2);x,y)$ of $P_N(M_1, M_2)$ is given by Equation 2.3.6 [**3**].

$$\begin{aligned}\bar{\chi}(P_N(M_1,M_2);x,y) = & \sum_{\text{flats W of N}} \frac{1}{x^{|W|}\chi(N/W;y)} \\ & \times \left(\sum_{\text{flats Z of N with W}\subseteq \text{Z}} \mu_N(W,Z)x^{|Z|}\bar{\chi}(M_1/Z;x,y) \right) \\ & \times \left(\sum_{\text{flats Z of N with W}\subseteq \text{Z}} \mu_N(W,Z)x^{|Z|}\bar{\chi}(M_2/Z;x,y) \right) \end{aligned} \tag{2.3.6}$$

where μ_N is the Möbius function of N. By applying Equations 2.2.2 and 2.3.2 to $\bar{\chi}(P_N(M_1, M_2);x,y)$ Bonin simplified Equation 2.3.6 to get Equation 2.3.7.

$$\begin{aligned}t(P_N(M_1,M_2);x,y) = & (y-1)^{r(T)} \sum_{\text{flats W of N}} \frac{1}{y^{|W|}\chi(N/W;(x-1)(y-1))} \\ & \times \left(\sum_{\text{flats Z of N with W}\subseteq \text{Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_1/Z;x,y) \right) \\ & \times \left(\sum_{\text{flats Z of N with W}\subseteq \text{Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_2/Z;x,y) \right) \end{aligned} \tag{2.3.7}$$

where μ_N is the Möbius function of N [**3**].

From the characterizations presented in this chapter, we can therefore use the same formula 2.3.7 to compute the Tutte polynomial of the cycle matroid $M(WS_n)$ of an n-wheel-sunflower, by substituting $M_1 = M(WS_n \setminus \{\{a,c\},\{c,b\}\})$ and $M_2 = M(W_n \cup \{\{a,c\},\{c,b\}\})$, where μ_N is the Möbius function of the common restriction N of M_1 and M_2.

CHAPTER 3

T-uniqueness of wheel-sunflowers

In this chapter we are going to show that wheel-sunflowers are T-unique as one of the applications of Tutte polynomials. A graph G is T-unique if any other graph H having the same Tutte polynomial as G is isomorphic to G. Several recent papers show that certain graphs such as wheels and ladders are determined by their Tutte polynomials [**4, 11, 12**].

The organization of this chapter is as follows. Section 3.1 is a review of basic properties of the Tutte polynomial as well as a summary of parameters of a graph G that are determined by its Tutte polynomial $t(G; x, y)$; Section 3.2 gives some properties of wheel-sunflower graphs which enable us to imitate the proofs of T-uniqueness of wheels. The first two sections contain main tools used in Section 3.3 to prove that wheel-sunflowers are T-unique.

3.1. The Tutte polynomial and rank-size generating polynomial

This section is a review of the definition of the rank-size *generating* polynomial in relation to the Tutte polynomial as found in [**12**]. We also include some parameters that are encoded in the Tutte polynomial of a graph G.

The rank-size generating polynomial is defined as: $F(G; x, y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|}$, where the coefficient of $x^i y^j$ in $F(G; x, y)$ counts the number of spanning subgraphs in G with rank i and j edges. Both $F(G; x, y)$ and $t(G; x, y)$ contain exactly the same information about G [**12**]. However, the Tutte polynomial $t(G; x, y)$ has several properties not shared by $F(G; x, y)$. In particular, $t(G; x, y)$ satisfies the fundamental contradiction rule $t(G; x, y) = t(G - e; x, y) + t(G/e; x, y)$, provided that $e \in E(G)$ is neither a coloop nor a loop, where $G - e$ and G/e denote the result of deleting and contracting the edge e in G respectively.

The proofs of Lemma 3.1.1, Theorems 3.1.2 and 3.1.3 are found in [**8**].

LEMMA 3.1.1. *Let* $t(G; x, y) = \sum b_{ij} x^i y^j$. *If* G *has neither coloops nor loops, then* $r(G) = max\{i : b_{i0} \neq 0\}$, $\quad n(G) = max\{j : b_{0j} \neq 0\}$.

Recall the definition of a 2-connected graph from Section 1.2.1.

THEOREM 3.1.2. *If $r(G)$ and $n(G)$ are both positive, then the number of 2-connected components of G is $min\{i : b_{i0} \neq 0\}$. Otherwise, G has $|E(G)|$ 2-connected components. In particular, If G is 2-connected and $t(H; x, y) = t(G; x, y)$, then H is also 2-connected.*

For the rest of this chapter we are going to follow the notation found in [**12**]. A cycle of length n is called C_n or an n-cycle; for $n = 3, 4, 5$ we also refer to them as triangles, squares and pentagons. A chord of cycle C is an edge joining two non-consecutive vertices of C. If $P(x, y)$ is a polynomial, we denote by $[x^i y^j]P(x, y)$ the coefficient of $x^i y^j$ in $P(x, y)$.

To prove that a graph is T-unique, we first show that some parameters of this graph are determined by its Tutte polynomial. Theorem 3.1.3 provides a list of such parameters which are used in this thesis. We refer to [**12**] for more details.

THEOREM 3.1.3. *Let $G = (V, E)$ be a 2-connected graph. Then the following parameters of G are determined by its Tutte polynomial.*

(i) The number of vertices and the number of edges.

(ii) The number of cycles of shortest length.

(iii) If G is simple, the number of cycles of length three, four and five; for cycles of length four it is also possible to determine how many of them have exactly one chord.

3.2. Properties of wheel-sunflower graphs

In this section we are going to give results on number of triangles, number of cycles of length four, each with exactly one chord, and number of cycles of length five, each with exactly two chords with a common vertex (that is triangulated pentagon), from wheel-sunflower graphs. These results will enable us to imitate the proof of T-uniqueness of wheels found in [**12**]. We start by presenting Lemma 3.2.1 which we will use to prove Proposition 3.2.2 on the number of triangles in an n-wheel-sunflower WS_n.

LEMMA 3.2.1. *Let $\tau(WS_n)$ be the number of triangles in an n-wheel-sunflower WS_n. Then $\tau(WS_{(n+1)}) = \tau(WS_n) + 2$, for $n \geq 2$.*

PROOF. From Definition 2.1.1, the construction of $WS_{(n+1)}$ from WS_n involves adding four edges to $E(WS_n)$: one rim edge, one spoke and two petal edges. These add two more triangles so that the total number of triangles in $WS_{(n+1)}$ becomes $\tau(WS_n) + 2$. □

PROPOSITION 3.2.2. *Let WS_n be an n-wheel-sunflower. Then $\tau(WS_n) = 2n$, for $n \geq 2$.*

PROOF. We are going to use induction on n. For $n = 2$, we get a 2-wheel-sunflower WS_2 which has 4 triangles. The two triangles have edge sets of a rim edge and two spokes $\{\{a_1, a_2\}, \{a_1, x\}, \{a_2, x\}\}, \{\{a_2, a_1\}, \{a_1, x\}, \{a_2, x\}\}$. And the remaining two have the edge sets of a rim edge and petal edges $\{\{a_1, a_2\}, \{a_1, c_1\}, \{a_2, c_1\}\}, \{\{a_1, a_2\}, \{a_1, c_2\}, \{a_2, c_2\}\}$. Assume the proposition holds for $n = k$ where $n > 2$. Then $\tau(WS_k) = 2k$. Consider the case $n = k+1$. Then by Lemma 3.2.1, we get $\tau(WS_{(k+1)}) = \tau(WS_k) + 2$. By the induction hypothesis we have $\tau(WS_{(k+1)}) = 2k + 2 = 2(k + 1)$. Hence the proposition holds by induction. □

Recall that a chord of a cycle is an edge joining two non-consecutive vertices of a cycle. Now we are going to give results on the number of all 4-cycles in an n-wheel-sunflower with exactly one chord.

LEMMA 3.2.3. *Let $\rho(WS_n)$ be the number of 4-cycles in WS_n with exactly one chord. Then $\rho(WS_{(n+1)}) = \rho(WS_n) + 2$, for $n \geq 2$.*

PROOF. To construct $WS_{(n+1)}$ from WS_n, using Lemma 3.2.1, we add four edges to $E(WS_n)$: one rim edge, one spoke and two petal edges. Each of the new rim edge and spoke contributes one chordal 4-cycle. So the total number of 4-cycles, with exactly one chord, in $WS_{(n+1)}$, is $\rho(WS_n) + 2$. □

PROPOSITION 3.2.4. *Let WS_n be an n-wheel-sunflower and $\rho(WS_n)$ be the number of 4-cycles with exactly one chord in WS_n. Then $\rho(WS_n) = 2n$ for $n \geq 3$.*

PROOF. We are going to use induction on n. For $n = 3$ we get a 3-wheel-sunflower WS_3 which has six 4-cycles with exactly one chord. Such six 4-cycles have the following edge sets. The first three edge sets have a rim edge as a chord. Since there are three rim edges in WS_n, then we have the following:

$$\{\{a_1, a_2\}, \{a_1, x\}, \{a_2, x\}, \{a_1, c_1\}, \{a_2, c_1\}\},$$

$$\{\{a_2, a_3\}, \{a_2, x\}, \{a_3, x\}, \{a_2, c_2\}, \{a_3, c_2\}\},$$

$$\{\{a_3, a_1\}, \{a_3, x\}, \{a_1, x\}, \{a_3, c_3\}, \{a_1, c_3\}\}.$$

In the next three edge sets the chord is a spoke. Since there are three spokes in WS_n, then we have the following:

$$\{\{a_1, x\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, x\}, \{x, a_2\}\},$$

$$\{\{a_2, x\}, \{a_2, a_3\}, \{a_3, a_1\}, \{a_1, x\}, \{x, a_3\}\},$$

$$\{\{a_3, x\}, \{a_3, a_1\}, \{a_1, a_2\}, \{a_2, x\}, \{x, a_1\}\}.$$

Assume the proposition holds for $n = k$ where $n > 3$. Then $\rho(WS_k) = 2k$. Consider the case $n = k+1$. Then by Lemma 3.2.3, we get $\rho(WS_{(k+1)}) = \rho(WS_k) + 2$. By the induction hypothesis we have $\rho(WS_{(k+1)}) = 2k+2 = 2(k+1)$. Therefore, the induction holds for any $n \geq 3$. □

Now we are going to give results on the number of all 5-cycles in an n-wheel-sunflower with exactly two chords with common vertex. Such type of 5-cycles are also called triangulated pentagons.

LEMMA 3.2.5. *Let $\zeta(WS_n)$ be the number of 5-cycles in WS_n with exactly two chords with a common vertex. Then $\zeta(WS_{(n+1)}) = \zeta(WS_n) + 3$, for $n \geq 4$.*

PROOF. To construct $WS_{(n+1)}$ from WS_n, using Lemma 3.2.1, we add four edges to $E(WS_n)$: one rim edge, one spoke and two petal edges. The new rim edge contributes contributes three triangulated pentagons: two of these include the new petal and one which involve spokes and rim edges only. So the total number of triangulated pentagons in $WS_{(n+1)}$ is $\zeta(WS_n) + 3$. □

In Proposition 3.2.6 we are excluding the cases of $n = 2, 3$ because WS_2 has no triangulated pentagons and a 3-wheel in a 3-wheel-sunflower has no triangulated pentagons.

PROPOSITION 3.2.6. *Let WS_n be an n-wheel-sunflower and $\zeta(WS_n)$ be the number of triangulated pentagons in WS_n. Then $\zeta(WS_n) = 3n$, for $n \geq 4$.*

PROOF. We are going to use induction on n. For $n = 4$, we get a 4-wheel-sunflower WS_4 which has twelve triangulated pentagons. Such twelve triangulated pentagons have the following edge sets. The first eight edge sets of triangualated pentagons have a petal contained in two as follows:

$$\{\{a_1, b_1\}, \{b_1, a_2\}, \{a_2, a_3\}, \{a_3, x\}, \{x, a_1\}, \{a_1, a_2\}, \{a_2, x\}\},$$

$$\{\{a_1, b_1\}, \{b_1, a_2\}, \{a_1, a_4\}, \{a_4, x\}, \{x, a_2\}, \{a_1, a_2\}, \{a_1, x\}\},$$

$$\{\{a_2, b_2\}, \{b_2, a_3\}, \{a_3, a_4\}, \{a_4, x\}, \{x, a_2\}, \{a_2, a_3\}, \{a_3, x\}\},$$

$$\{\{a_2, b_2\}, \{b_2, a_3\}, \{a_2, a_1\}, \{a_1, x\}, \{x, a_3\}, \{a_3, a_2\}, \{a_2, x\}\},$$

$$\{\{a_3, b_3\}, \{b_3, a_4\}, \{a_4, a_1\}, \{a_1, x\}, \{x, a_3\}, \{a_3, a_4\}, \{a_4, x\}\},$$

$$\{\{a_3, b_3\}, \{b_3, a_4\}, \{a_3, a_2\}, \{a_2, x\}, \{x, a_4\}, \{a_4, a_3\}, \{a_3, x\}\},$$

$$\{\{a_4, b_4\}, \{b_4, a_1\}, \{a_1, a_2\}, \{a_2, x\}, \{x, a_4\}, \{a_4, a_1\}, \{a_1, x\}\},$$

$$\{\{a_4, b_4\}, \{b_4, a_1\}, \{a_4, a_3\}, \{a_3, x\}, \{x, a_1\}, \{a_1, a_4\}, \{a_4, x\}\}.$$

Each of the last four edge sets involves four spokes and three rim edges only as follows:

$$\{\{a_1, x\}, \{a_2, x\}, \{a_3, x\}, \{a_4, x\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}\},$$

$$\{\{a_2, x\}, \{a_3, x\}, \{a_4, x\}, \{a_1, x\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\},$$

$$\{\{a_3, x\}, \{a_4, x\}, \{a_1, x\}, \{a_2, x\}, \{a_3, a_4\}, \{a_4, a_1\}, \{a_1, a_2\}\},$$

$$\{\{a_4, x\}, \{a_1, x\}, \{a_2, x\}, \{a_3, x\}, \{a_4, a_1\}, \{a_1, a_2\}, \{a_2, a_3\}\}.$$

Assume the proposition holds for $n = k$ where $n > 4$. Then $\zeta(WS_k) = 3k$. Consider the case $n = k + 1$. Then by Lemma 3.2.5, we get $\zeta(WS_{(k+1)}) = \zeta(WS_k) + 3$. By the induction hypothesis we have: $\zeta(WS_{(k+1)}) = 3k + 3 = 3(k + 1)$. Therefore, the induction holds for any $n \geq 4$. □

3.3. T-uniqueness

Our aim in this section is to study which information about an n-wheel-sunflower graph WS_n, is contained in its Tutte polynomial and use it to prove its T-uniqueness.

In Proposition 3.3.1 we are going to prove T-uniqueness of wheel-sunflowers by imitating the routine proof used to prove T-uniqueness of wheels [**12**]. The technique requires that we consider the case where the smallest circuit of a graph G is a triangle and an edge in G does not belong to three or more triangles. Because of this we are going to exclude WS_n where $n = 2, 3$.

PROPOSITION 3.3.1. *For every $n \geq 4$, the wheel-sunflower WS_n is T-unique.*

PROOF. We begin our proof by taking a graph H such that $t(H;x,y) = t(WS_n;x,y)$ for $n \geq 4$. Then by applying Theorem 3.1.2 we get that H is a 2-connected graph. Since the graph H is 2-connected and $t(H;x,y) = t(WS_n;x,y)$, then by applying Theorem 3.1.3 to H we also obtain the following additional information:

(i) $|V(H)| = |V(WS_n)| = 2n+1$, from Definition 2.1.1 of WS_n;
(ii) $|E(H)| = |E(WS_n)| = 4n$, from Proposition 2.1.3 of Section 2.1.1;
(iii) H has $2n$ triangles, as WS_n, in Proposition 3.2.2;
(iv) H has $2n$ cycles of length 4, as WS_n, all of which have exactly one chord, from Proposition 3.2.4.

Now before proving T-uniqueness of wheel-sunflower graphs for $n \geq 4$, we are going to prove a couple of claims that must be satisfied by the chosen graph H. We start by reviewing the definitions of a complete graph and a bipartite graph as found in [**20**].

A complete graph is a graph where every pair of distinct vertices are adjacent. A complete graph on n-vertices is denoted by K_n. A bipartite graph is a graph whose vertex set can be partitioned into two non empty sets V_1 and V_2 in such a way that every edge joins a vertex in V_1 to a vertex in V_2. A complete bipartite graph has every vertex in V_1 adjacent to every vertex in V_2. If V_1 has r vertices and V_2 has s vertices then the complete bipartite graph is written as $K_{r,s}$ and V_1 and V_2 are called partite sets of size r and s respectively.

Now we are going to state and prove the claims. Claim 1 entirely corresponds to that of T-uniqueness of wheels found in [**12**]. We prefer to repeat the same here for completeness of T-uniqueness of wheel-sunflower graphs.

Claim 1. There is no edge in $E(H)$ belonging to three or more triangles.

PROOF. Let us show that it is possible to obtain from $t(H;x,y) = t(WS_n;x,y)$ the number of subsets of $E(H)$ consisting of three triangles meeting at a single edge, or which is the same, or subgraphs $K_{2,3}$ with an extra edge joining the two vertices in the partite set of cardinality two: call these subsets $K^+_{2,3}$. Such type of subgraphs contribute to the coefficient of x^4y^7 in $F(H;x,y)$. There are other three subsets of $E(H)$ that contribute to the coefficient of x^4y^7. We study first which other subsets of $E(H)$ contribute to this coefficient (Figure 3.1):

(1) Cycles of length five with two chords with a common vertex (that is triangulated pentagon).

(2) Subgraphs $K_{2,3}$ with an extra edge joining two vertices in the partite set of order three.

(3) Complete subgraphs K_4 plus one edge.

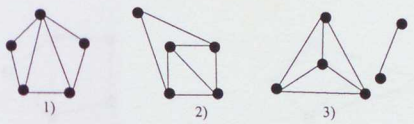

FIGURE 3.1. All possible subgraphs in H with rank 4 and size 7 in addition to $K_{2,3}^{+}$

The situation described in (3) is impossible because there are no K_4 in H. Using the fact that each cycle of length four must have one and only one chord then situation (2) is also impossible. This implies that a cycle of length 5 cannot have three or more chords. Therefore, the only contributions to $[x^4y^3]F(G;x,y)$ are $K_{2,3}^{+}$ and triangulated pentagons to get Equation 3.3.1:

$$[x^4y^3]F(G;x,y) \;=\; t+c \tag{3.3.1}$$

where c is the number of cycles of length five as in equation 3.3.1 and t is the number of $K_{2,3}^{+}$ [**12**]. Now we have to show that the value of t in equation 3.3.1 is zero. To see that $t=0$ we need another equation involving t and c. This equation is obtained by analyzing the coefficient of x^4y^6 in $F(H;x,y)$; that is, subgraphs in H with rank 4 and size 6. The possibilities are the following (Figure 3.2):

(1) Two edge-disjoint triangles.

(2) A cycle of length four with its chord and any other edge.

(3) A cycle of length five with one chord.

(4) A subgraph $K_{2,3}$.

Such type of subgraphs presented in (1) and (2) contribute $\frac{n(n-1)}{2\times 1} - n$ and $n(2n-5)$, respectively, to the coefficients in $F(H;x,y)$, where n is the order of each subgraph. Each cycle of length five must be triangulated, otherwise there would be a cycle of length four without chords. Since the maximum number of chords in a cycle of length five is two, see Figure 3.1 case (1), then the number of cycles of length five with one chord equals twice the number of triangulated pentagons. On the other hand, in $K_{2,3}$ there are three cycles of length four without chords, refer to Figure 3.2 case (4) because that is $K_{2,3}$. As we have

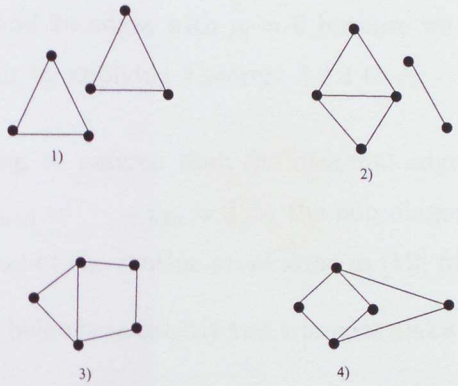

FIGURE 3.2. All possible subgraphs in H with rank 4 and size 6

seen previously the chord must join the two vertices in the partite set of cardinality two. This means that in H the number of $K_{2,3}$ equals the number of $K^{+}_{2,3}$. Hence we obtain Equation 3.3.2 involving t and c.

$$[x^4y^2]F(G;x,y) = t + 2c + \frac{n(n-1)}{2\times 1} - n + n(2n-5). \tag{3.3.2}$$

Equations 3.3.1 and 3.3.2 imply that it is possible to obtain t and c from the coefficients of $t(H;x,y) = t(WS_n;x,y)$. As in WS_n, for $n \geq 4$, there is no edge belonging to three triangles, then the value of t must be equal to zero. □

The proof of the next claim is an imitation of the proof found in [**12**] for wheels. We start with some definitions and notation used in Claim 2. We are going to call an edge $e \in E(H)$ a *diagonal* edge if it is the chord of one cycle of length four; if e is the chord of the 4-cycle C_4, then we write $e = d(C_4)$. If e is not the chord of any cycle of length 4, we are going to call it a *non-diagonal* edge. In a wheel-sunflower graph, diagonal edges are rim edges and spokes, while the non-diagonal edges are petal edges.

Claim 2. In H there are $2n$ diagonal edges and $2n$ non-diagonal edges.

PROOF. Let U be the set $U = \{(e, C_4) : e = d(C_4)\}$. Let us define ρ_i as the number of cycles of length 4 having an edge e_i as a chord. By Claim 1 we know that $0 \leq \rho_i \leq 1$, that is each cycle of length 4 has either one chord or zero. Therefore the cardinality of U equals $2n$ because there are exactly $2n$ cycles of length 4, from Proposition 3.2.4. But also we know that there are $4n$ edges in WS_n, from Proposition 2.1.3 of Section 2.1.1. Assume that every edge in WS_n is a chord. Then $|U| = \rho_1 + \cdots + \rho_{4n}$. This implies that there must

be $2n$ edges with $\rho_i = 1$ and $2n$ edges with $\rho_i = 0$ because we have already said that H has $2n$-cycles of length four by applying Theorem 3.2.4 to H. □

From now on we are going to assume that the diagonal edges are $e_1, \cdots, e_{2n}$; that is, $\rho_1 = \cdots = \rho_{2n} = 1$ and $\rho_{2n+1} = \cdots = \rho_{4n} = 0$ for the non-diagonal edges $\{e_{2n+1}, \cdots, e_{4n}\}$. Claim 3 is also an imitation of the routine proof done in [**12**] for wheels.

Claim 3. A diagonal edge belongs to exactly two triangles and a non-diagonal edge belongs to exactly one.

PROOF. Consider the set $F = \{(e,t) : e \text{ is an edge of triangle } t\}$. Let us define τ_i as the number of triangles that contain an edge e_i. From Claims 1 and 2 we know that $\tau_i = 2$ for $1 \leq i \leq 2n$ and that $\tau_i \leq 1$ for $2n+1 \leq i \leq 4n$. Therefore summing over all triangles in H, we deduce that the maximum value of $|F|$ is $2n + 4n = 6n$. But by summing over all $4n$ edges, we obtain $|F| = \tau_1 + \cdots + \tau_{4n} = 4n + \tau_{2n+1} + \cdots + \tau_{4n} \leq 6n$. Hence $\tau_i = 1$ for $2n+1 \leq i \leq 4n$ and the claim is proved. □

Now in an n-wheel-sunflower graph let us call all squares, whose diagonal edges are rim edges, as diamonds and denote them by d_i. So we define a diamond d_i as follows.

DEFINITION 3.3.2. A diamond d_i, in an n-wheel-sunflower graph is a square having a rim edge as a chord.

Claim 4. If every vertex $v \in V(H)$ is incident with a non-diagonal edge, then there is a second non-adjacent edge incident with v.

PROOF. Let f be the non-diagonal edge incident with v and let d_1 be the unique diamond that contains f. Then this diamond d_1 contains another edge incident with v, call it f_1. If f_1 is a non-diagonal edge, we have finished. If not, then f_1 is a diagonal edge and there exists a diamond d_2 different from d_1 and containing f_1. Let f_2 be the second edge of d_2 incident with v. Then by applying repeatedly the previous argument, we construct a succession of diagonal edges $f, f_1, f_2, \cdots, f_k$ incident with v. Note that edges are not repeated in this list, because, by Claim 1, there is no edge belonging to three triangles. Furthermore, note that these diagonal edges correspond to spokes in an n-wheel-sunflower graph. As the number of edges is finite, we must necessarily end in a non-diagonal edge incident with v, necessarily different from f_1. □

Now to prove the next claim we are going to use the following definitions. A graph is connected if every pair of vertices is joined by a path. Secondly, a maximally connected component is a connected subgraph of a graph to which no vertex can be added and it still be connected. This maximally connected component will be referred to as the connected component in this thesis. Hence, examples of connected components in WS_n are wheel-sunflowers with a missing spoke $WS_n \setminus \{a, x\}$ or wheel-sunflowers with a missing rim edge $WS_n \setminus \{a, b\}$. Their rank is still $2n$ and the size is $4n - 1$ since they have one edge missing. For a thorough introduction to the subject of connected graphs we refer to [**20**]. The following Lemma is used in proving Claim 5.

LEMMA 3.3.3. *Let $WS_n \setminus \{\{a_i, x\}\}$ be an n-wheel-sunflower graph without spokes $\{a_i, x\}$. Then $|E(WS_n \setminus \{\{a_i, x\}\})| = 3n$.*

PROOF. We know that an n-wheel-sunflower WS_n has $4n$ edges from Proposition 2.1.3 of Section 2.1.1. Furthermore, from Definition 2.1.1 of WS_n there are n spokes. Since $WS_n \setminus \{\{a_i, x\}\}$ is obtained by deleting all spokes $\{a_i, x\}$ in WS_n, then $|E(WS_n \setminus \{\{a_i, x\}\})| = |E(WS_n)| - |\bigcup_{i=1}^{n} \{a_i, x\}| = 4n - n$. □

Claim 5. There is no subgraph in H isomorphic to the wheel-sunflower WS_p for $p < n$.

PROOF. Recall the relationship between rank and size of the wheel-sunflower. We know that WS_p has rank $2p$ from Section 2.1.2 and size $4p$ from Proposition 2.1.3 of Section 2.1.1.

The proof of Claim 5 consists of showing that the coefficient of $x^{2p}y^{4p}$ in $F(H; x, y)$ is zero for all $p < n$. This is equivalent to proving that in WS_n there is no subgraph G of rank $2p$ and size $4p$. We are going to use this equivalent form. Then we are going to show that it is impossible to find a subgraph in WS_n with the relationship of rank $2p$ and size $4p$.

If a subgraph of WS_n does not contain the central vertex (the hub), then we have the following possible cases:

(i) it is a forest or;
(ii) edge-disjoint or vertex-joint triangles or;
(iii) a graph $WS_n \setminus \{\{a_i, x\}\}$ and its size is $3n$ from Lemma 3.3.3 and its rank is $|V(WS_n \setminus \{\{a_i, x\}\})| - 1 = 2n - 1$, so it cannot be a wheel-sunflower WS_n.

Therefore, the subgraph G must contain the central vertex. Now we are going to consider the subgraph that is the maximally connected component in WS_k that contains the central vertex. Let $WS_k \setminus \{a, x\}$ be an n-wheel-sunflower with a missing spoke $\{a, x\}$. This is the connected component in WS_k that contains the central vertex x with rank $2k$ which is the same as that of WS_k but its size is $4k-1$. The same argument applies to another connected component of WS_k which is an n-wheel-sunflower with a missing rim edge $\{a, b\}$ denoted by $WS_k \setminus \{a, b\}$. This implies that the connected component G that contains this central vertex has rank $2k$ and size at most $4k-1$. The rest of the components are edge-disjoint or vertex-joint triangles and forests. So again it is impossible to reach the required relation between rank and size, which is rank $2p$ and size $4p$. □

Recall Proposition 3.3.1 about T-uniqueness of wheel-sunflowers which we re-state and prove now.

PROPOSITION 3.3.4. *For every $n \geq 4$, the wheel-sunflower WS_n is T-unique.*

PROOF. To prove this it means we should be able to construct an n-wheel-sunflower, for $n \geq 4$, isomorphic to a graph H, using the diamonds d_i which we have defined in Definition 3.3.2. Recall that the set of all non-diagonal edges is denoted by $\{e_{2n+1}, \cdots, e_{4n}\}$. Now let $\mathcal{V}$ be the set of all elements of the form (e, v) where $e \in \{e_{2n+1}, \cdots, e_{4n}\}$ is a non-diagonal edge in G and v is a vertex incident with e in G such that G is a subgraph of H. Recall that non-diagonal edges correspond to petal edges in WS_n. Therefore the size $|\mathcal{V}| = 6n - 2n$ because there are $2n$ non-diagonal edges, each non-diagonal edge is incident to two vertices and half of the number of vertices is counted twice. Let δ'_i be the non-diagonal degree of v_i, that is, the number of non-diagonal edges incident with v_i. By Claim 4, either δ'_i is zero or is greater than one. Now recall that H has $2n+1$ vertices. Therefore, by using all the vertices of H, we must have:

$$|\mathcal{V}| = \delta'_1 + \delta'_2 + \cdots + \delta'_{2n+1} = 4n.$$

This implies that there is at least one vertex with non-diagonal degree equal to zero; assume it is v_1. That means v_1 is incident only with diagonal edges. We can now reconstruct the wheel-sunflower WS_n as follows. Let f_1 be an edge incident with v_1; then f_1 belongs to a diamond d_1 which contains a second edge, f_2, incident with v_1. As f_2 is diagonal, there exists a new diamond d_2 containing f_2 and an edge f_3 incident with v_1. Applying successively the same argument, we obtain a list of diagonal edges $f_1, \cdots, f_k$ incident with

CHAPTER 4

Component numbers of links from wheel-sunflowers

In this chapter we apply the theory of Tutte polynomials to classify links associated with wheel-sunflower graphs using component numbers. We start by reviewing the relationship between a graph and its corresponding link in Section 4.1 and use this relationship to construct links from wheel-sunflower graphs. Then we give number of components in links corresponding to wheel-sunflowers and their variations in Sections 4.2 and 4.2.1.

4.1. Construction of a link from an n-wheel-sunflower

In this section we introduce how a link $K(WS_n)$ from an n-wheel-sunflower WS_n is constructed. To do this we are going to construct a diagram which we call the link universe $U(K(WS_n))$ of an n-wheel-sunflower.

We start by constructing the *medial* graph from an n-wheel-sunflower and then replace the vertices of the medial graph by crossings. Then we define the part which goes under or over at a crossing to get a link $K(WS_n)$. The medial graph of a graph WS_n denoted as $H = H(WS_n)$ is defined as follows:

DEFINITION 4.1.1. Let the vertices of H correspond to edges $E(WS_n)$ of WS_n and let two vertices of H be connected by an edge if the corresponding edges in WS_n are subsequent in the cyclic order of edges around some vertex in WS_n. Then $H = H(WS_n)$ is called the medial graph of WS_n.

By replacing the vertices of $H(WS_n)$ with *crossings* the resulting diagram is called the *link universe* corresponding to WS_n and is denoted by $U(K(WS_n))$. It is called the link universe because it does not give information about the crossings - that is, it does not say which crossing is an overpass and which one is an underpass [**15**].

EXAMPLE 4.1.2. Figure 4.1 shows a link universe corresponding to a 6-wheel-sunflower. Notice that the dotted graph is a 6-wheel-sunflower WS_6, while the solid lines form its corresponding link universe.

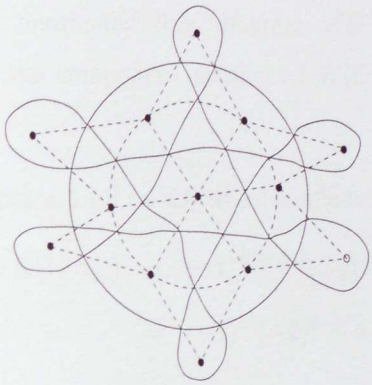

FIGURE 4.1. A link universe from a 6-wheel-sunflower

A link $K(WS_n)$ from WS_n is obtained from a link universe by defining the part which goes under or over at a crossing. Figure 4.2 shows a construction of a 6-wheel-sunflower (dotted lines) and its corresponding link, $K(WS_6)$ (solid lines).

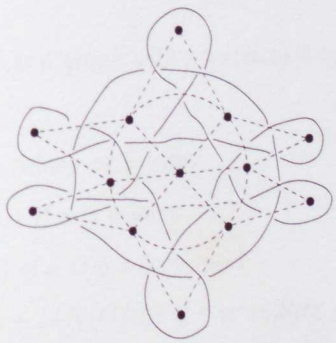

FIGURE 4.2. A link $K(WS_6)$ corresponding to WS_6

Notice that by tracing crossing cycles in the link diagram $K(WS_6)$, there are three *components*. The term *component* means a crossing cycle in a link and the term *component number* means the number of crossing cycles that form a link. For further details on construction of links from graphs we refer the reader to [**14**].

In this thesis we are going to consider the case where $K(WS_n)$ is an *unoriented* link diagram corresponding to n-wheel-sunflowers WS_n. That is, we are going to consider the case of links without direction. In this case the linking number of $K(G)$, denoted as $L(K(G))$ is defined as follows [**15**]:

DEFINITION 4.1.3. For an unoriented link diagram $K(G)$, a number $L(K(G)) = (-1)^{E(G)}(-2)^{c-1}$ where c is the component number of $K(G)$ and $E(G)$ is the edge set of G.

EXAMPLE 4.1.4. Using Defintion 4.1.3 on a 6-wheel-sunflower, WS_6, we have

$$\begin{aligned} L(K(WS_6)) &= (-1)^{E(WS_6)}(-2)^{c-1} \\ &= (-1)^{24}(-2)^{3-1} = 4. \end{aligned}$$

For a graph G, with the Tutte polynomial, $t(G, x, y)$, it is well known that the evaluation $t(G, -1, -1) = (-1)^{E(G)}(-2)^{c-1}$, where c is the component number of the link whose universe is the medial graph of G [**15**].

Now we review relevant ideas playing a role of determining linking numbers in link diagrams associated wheel-sunflowers. Proposition 4.1.5 is given by Mphako [**15**].

PROPOSITION 4.1.5. *Let G be a planar graph. Then*

(i) $L(K(G)) = (-2)^{n-1}$ *if G is a graph with n vertices and no edges. Thus* $L(K(G)) = 1$ *if G is a single vertex;*

(ii) $L(K(G)) = -1$ *if G is a coloop or a loop;*

(iii) $L(K(G)) = -L(K(G \setminus e))$ *if e is a loop;*

(iv) $L(K(G)) = -L(K(G/e))$ *if e is a coloop; and*

(v) $L(K(G)) = L(K(G \setminus e)) + L(K(G/e))$ *if e is neither a loop nor a coloop.*

The next proposition, also given by Mphako [**15**], gives other graph operations which do not affect the component number of the associated link. Proposition 4.1.7, uses the following definitions.

DEFINITION 4.1.6. For any graph G :

(i) A series pair is a pair of any two edges e and f in G whose common vertex has degree 2.

(ii) A parallel pair is a pair of any two edges e and f that form a 2-cycle in G.

PROPOSITION 4.1.7. *Let G be a planar graph. Then*

(i) $L(K(G)) = L(K(G/e/f))$ *if e and f are a series pair; and*

(ii) $L(K(G)) = L(K(G \setminus e \setminus f))$ *if e and f are a parallel pair.*

4.2. Component numbers of links associated with wheel-sunflowers

In this section we determine component numbers of links corresponding to wheel-sunflower graphs. We start by looking at how these component numbers change with the removal of certain edges in an n-wheel-sunflower graph.

The following notation is used. A spoke is denoted by $\{a_i, x\}$, where a_i is any vertex in the rim and x is the central vertex of an n-wheel-sunflower. Recall that an edge of the form $\{a_i, a_{i+1}\}$ for all $i \in \{1, 2, \cdots, n, n+1 = 1\}$ in an n-wheel-sunflower is called a rim edge. But from now on, if one rim edge is under consideration we will denote it as $\{a, b\}$. As such we are going to call an edge set of the form $\{\{a, c\}, \{c, b\}\}$ as a petal such that c is the vertex where the two petal edges $\{a, c\}$, $\{c, b\}$ intersect and $\{a, b\}$ is a rim edge under consideration in WS_n. Lemma 4.2.1 is used in Proposition 4.2.2.

LEMMA 4.2.1. *Let $WS_3 \setminus \{a, x\}$ be a 3-wheel-sunflower with a missing spoke $\{a, x\}$ and $K(WS_3 \setminus \{a, x\})$ be its corresponding link. Then $K(WS_3 \setminus \{a, x\})$ has three components.*

PROOF. By using the method in Section 4.1 we construct $K(WS_3 \setminus \{a, x\})$ from $WS_3 \setminus \{a, x\}$ in Figure 4.3 and count its components. □

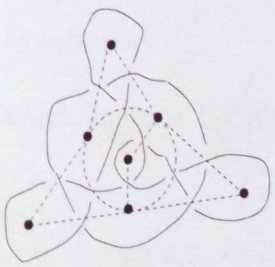

FIGURE 4.3. A link $K(WS_3 \setminus \{a, x\})$ corresponding to $WS_3 \setminus \{a, x\}$

PROPOSITION 4.2.2. *Let $WS_n \setminus \{a, x\}$ be an n-wheel-sunflower with a missing spoke $\{a, x\}$ and $K(WS_n \setminus \{a, x\})$ be its corresponding link. Then*

(i) If n is odd and $n \geq 3$ then $K(WS_n \setminus \{a, x\})$ is a 3-link.
(ii) If n is even and $n \geq 2$ then $K(WS_n \setminus \{a, x\})$ is a 2-link.

PROOF. We use induction on n. If $n = 2$ then $K(WS_2 \setminus \{a, x\})$ is a link 7_1^2 from the table of knots in the Appendix. Hence it has two components. For $n = 3$ we have $L(K(WS_3 \setminus \{a, x\}))$ which is a 3-link by Lemma 4.2.1. Assume it is true for $n = 2m - 1$

and $n = 2m$ for some $m > 1$. Now consider $n = 2m+1$ and $n = 2m+2$. Assume that each diagram in Figure 4.4 represents the number L of the link corresponding to the diagram. We start with a d-wheel-sunflower with a missing spoke $\{a, x\}$ denoted by $WS_d \setminus \{a, x\}$ and increase it to a $(d+2)$-wheel-sunflower with a missing spoke $WS_{(d+2)} \setminus \{a, x\}$. Thus by applying Propositions 4.1.7(i), and 4.1.5(iii) accordingly we get Figure 4.4.

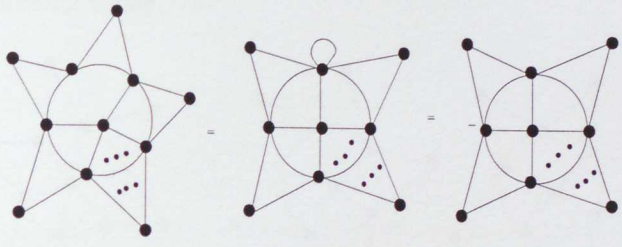

FIGURE 4.4. Computation of $L(K(WS_d \setminus a, x))$

Hence $L(K(WS_{(d+2)} \setminus \{a, x\})) = -L(K(WS_d \setminus \{a, x\}))$. Thus substituting $d+2$ by $n = 2m+1$ or $n = 2m+2$ we have the following results:

$$L(K(WS_{(2m+1)} \setminus \{a, x\})) = L(K(WS_{(2m-1)} \setminus \{a, x\})),$$
$$L(K(WS_{(2m+2)} \setminus \{a, x\})) = L(K(WS_{(2m)} \setminus \{a, x\})).$$

Therefore, the proposition holds by induction. □

LEMMA 4.2.3. *Let WS_2 be a 2-wheel-sunflower and $K(WS_2)$ be its corresponding link. Then $K(WS_2)$ has 3 components.*

PROOF. By using the method in Section 4.1, we construct $K(WS_2)$ from WS_2 in Figure 4.5 and count its components. □

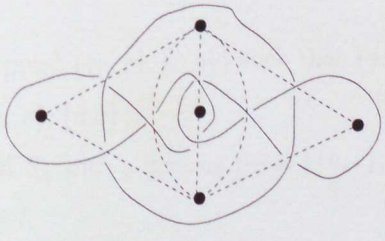

FIGURE 4.5. A link $K(WS_2)$ corresponding to WS_2

PROPOSITION 4.2.4. *Let WS_n be an n-wheel-sunflower and $K(WS_n)$ be its corresponding link. Then $K(WS_n)$ is a 3-link if n is even. Otherwise it is a 2-link.*

PROOF. By induction on n. If $n = 2$ we have $L(K(WS_2))$, which is a 3-link by Lemma 4.2.3. Assume it is true for $n = 2m - 1$ and $2m$ for some $m > 1$. Now consider $n = 2m + 1$ and $2m + 2$. Assume that each diagram in Figure 4.6 represents the number L of the link corresponding to the diagram. If we start with a d-wheel-sunflower then by applying Propositions 4.1.7(i), 4.1.7(ii) and 4.1.5(iii) accordingly, we get a $(d-1)$-wheel-sunflower with a missing spoke.

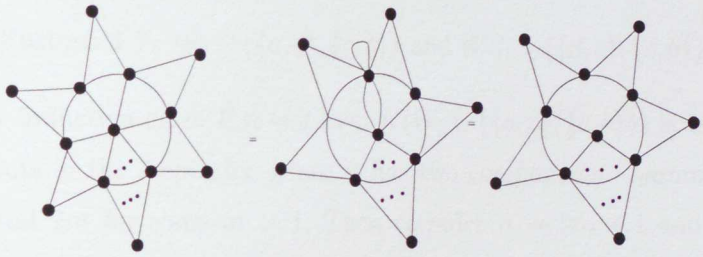

FIGURE 4.6. Computation of $L(K(WS_d))$

If $d = 2m + 1$ then by Proposition 4.2.2(ii), it is a 2-link. Similarly, if $d = 2m$ then by Proposition 4.2.2(i), it is a 3-link. Therefore, by induction the proposition holds. □

Proposition 4.2.5 on wheels is given by Mphako [**15**] and we will need it to prove Proposition 4.2.6.

PROPOSITION 4.2.5. *Let $W_n \setminus \{a, x\}$ be an n-wheel with a missing spoke $\{a, x\}$, where x is the central vertex of W_n, and $K(W_n \setminus \{a, x\})$ be its corresponding link. Then $K(W_n \setminus \{a, x\})$ is a 2-link.*

Recall from Chapter 2, Section 2.3, that:

(i) An n-wheel W_n with a petal $\{\{a, c\}, \{c, b\}\}$ such that $\{a, b\} \in E(W_n)$ and $c \notin V(W_n)$ is denoted by $W_n \cup \{\{a, c\}, \{c, b\}\}$;

(ii) An n-wheel-sunflower WS_n with a missing petal $\{\{a, c\}, \{c, b\}\}$ is denoted by $WS_n \setminus \{\{a, c\}, \{c, b\}\}$ Figure 4.7.

PROPOSITION 4.2.6. *Let $W_n \cup \{\{a, c\}, \{c, b\}\}$ be an n-wheel with a petal $\{\{a, c\}, \{c, b\}\}$, such that $\{a, b\}$ is a rim edge and $K(W_n \cup \{\{a, c\}, \{c, b\}\})$ be its corresponding link. Then $K(W_n \cup \{\{a, c\}, \{c, b\}\})$ is a 2-link.*

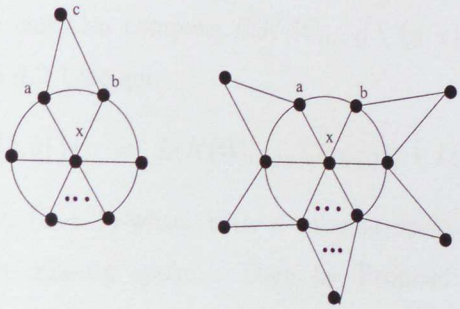

FIGURE 4.7. $W_n \cup \{\{a,c\},\{c,b\}\}$ and $WS_n \setminus \{\{a,c\},\{c,b\}\}$

PROOF. By induction on n. If $n = 2$ then $K(W_2 \cup \{\{a,c\},\{c,b\}\})$ is the link 6_1^2 from the table of knots in the Appendix, hence it has two components. Assume it is true for $n = 2m-1$ and $2m$ for some $m > 1$. Then consider $n = 2m+1$ and $n = 2m+2$. Then we trace the rim edge $\{a,b\}$ on an n-wheel such that $\{\{a,c\},\{c,b\}\}$ is a petal. Thus by applying Proposition 4.1.5(v) on an edge $\{a,b\}$ in $W_n \cup \{\{a,c\},\{c,b\}\}$, we have Equation 4.2.1 from Figure 4.8:

$$\begin{aligned} L(K(W_n \cup \{\{a,c\},\{c,b\}\})) &= L(K(W_n \cup \{\{a,c\},\{c,b\}\} \setminus \{a,b\})) \\ &+ L(K(W_n \cup \{\{a,c\},\{c,b\}\}/\{a,b\})). \end{aligned} \tag{4.2.1}$$

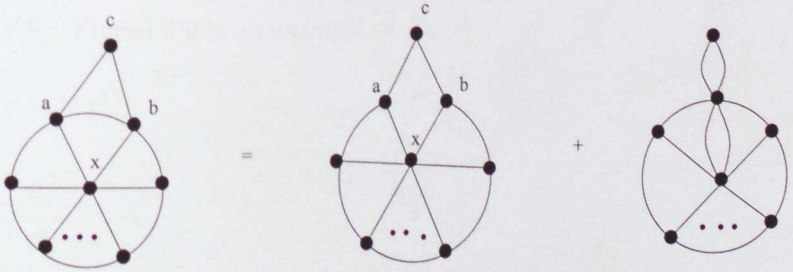

FIGURE 4.8. Computation of $L(K(W_n \cup \{\{a,c\},\{c,b\}\}))$

But $(W_n \cup \{\{a,c\},\{c,b\}\}) \setminus \{a,b\}$ is an $(n+1)$-wheel with a missing spoke $\{c,x\}$, which we denote as $W_{(n+1)} \setminus \{c,x\}$ and we can easily compute $L(K(W_{(n+1)} \setminus \{c,x\}))$ by Proposition 4.2.5. Furthermore $L(K(W_n \cup \{\{a,c\},\{c,b\}\}/\{a,b\}))$ can be simplified by deleting the two parallel pairs of edges by Proposition 4.1.7(ii). We thus get an $(n-1)$-wheel with a missing spoke, which we denote as $W_{(n-1)} \setminus \{a,x\}$ in the next equation.

$$L(K((W_n \cup \{\{a,c\},\{c,b\}\})/\{a,b\})) = L(K(W_{(n-1)} \setminus \{a,x\})).$$

By Proposition 4.2.5 we can also compute $L(K(W_{(n-1)} \setminus \{a, x\}))$. Thus by substituting these back into Equation 4.2.1 we get:

$$L(K(W_n \cup \{\{a, c\}, \{c, b\}\})) = L(K(W_{(n+1)} \setminus \{c, x\})) + L(K(W_{(n-1)} \setminus \{a, x\})).$$

Let $W_{(d+1)} \setminus \{c, x\}$ be a $(d+1)$-wheel with a missing spoke and $W_{(d-1)} \setminus \{a, x\}$ be a $(d-1)$-wheel with a missing spoke. Then by Proposition 4.2.5 we know that $L(K(W_{(d+1)} \setminus \{c, x\})) = L(K(W_{(d-1)} \setminus \{a, x\}))$. Hence $L(K(W_d \cup \{\{a, c\}, \{c, b\}\})) = L(K(W_{(d-2)} \cup \{\{a, c\}, \{c, b\}\}))$. Thus for $d = 2m+1$ and $2m+2$ we have:

$$L(K(W_{(2m+1)} \cup \{\{a, c\}, \{c, b\}\})) = L(K(W_{(2m-1)} \cup \{\{a, c\}, \{c, b\}\})),$$

$$L(K(W_{(2m+2)} \cup \{\{a, c\}, \{c, b\}\})) = L(K(W_{(2m)} \cup \{\{a, c\}, \{c, b\}\})).$$

Therefore, by induction the proposition holds. □

4.2.1. Component numbers of links associated with F_n.

In this sub-section we are going to introduce a class of n-fan-sunflower graphs as one of the variations of an n-wheel-sunflower and then determine the component number of any link whose associated graph is in this class. We are going to denote an n-fan-sunflower graph by F_n.

An n-fan-sunflower F_n is obtained by deleting one "outer" trianlge from an n-wheel-sunflower WS_n. Figure 4.9 is an example of F_6.

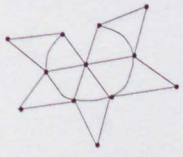

FIGURE 4.9. A 6-fan-sunflower F_6

LEMMA 4.2.7. *$K(F_3)$ has one component.*

PROOF. By using the method in Section 4.1, we construct $K(F_3)$ from F_3 in Figure 4.10 and count the number of components. □

PROPOSITION 4.2.8. *Let F_n be an n-fan-sunflower and $K(F_n)$ be its associated link. Then $K(F_n)$ is a 2-link if n is even where $n \geq 2$. And if n is odd where $n \geq 3$ then $K(F_n)$ is a knot.*

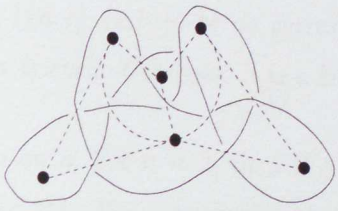

FIGURE 4.10. F_3 and its corresponding Link $K(F_3)$

PROOF. By induction on n. If $n = 2$, then $K(F_2)$ is a link 5_1^2 which has two components from the table of knots in the Appendix. Furthermore, if $n = 3$ then $K(F_3)$ has one component from Lemma 4.2.7. Therefore, the proposition is true for $n = 2, 3$. Assume it is true for $n = 2m - 1$ and $2m$ for some $m > 1$. Then we are going to show that it is also true for $n = 2m + 1$ and $2m + 2$. We start with a d-fan-sunflower, D say, and increase it to a $(d + 2)$-fan-sunflower. Assume that each diagram in the figure represents the number $L(K(F_n))$ of the diagram. Thus by applying Propositions 4.1.7(i), 4.1.7(ii) and 4.1.5(iii) accordingly, we get Figure 4.11.

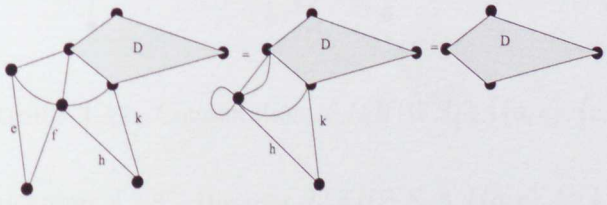

FIGURE 4.11. Computation of $L(K(F_d))$

Hence $L(K(F_{(d+2)})) = L(K(F_d))$. Thus substituting $(d + 2)$ by n we have the following equations:

$$L(K(F_{(2m+1)})) = L(K(F_{(2m-1)})),$$
$$L(K(F_{(2m+2)})) = L(K(F_{(2m)})).$$

Therefore, the proposition holds by induction. □

PROPOSITION 4.2.9. *Let $WS_n \setminus \{\{a,c\},\{c,b\}\}$ be an n-wheel-sunflower with a missing petal $\{\{a,c\},\{c,b\}\}$ and $K(WS_n \setminus \{\{a,c\},\{c,b\}\})$ be its corresponding link. Then $K(WS_n \setminus \{\{a,c\},\{c,b\}\})$ is a 2-link if n is even. Otherwise it is a knot.*

PROOF. We use induction on n. For $n = 2$, then $WS_2 \setminus \{\{a,c\},\{c,b\}\}$ is a 2-wheel with one petal $W_2 \cup \{\{a,c\},\{c,b\}\}$. Hence by Proposition 4.2.6, we know that $K(W_2 \cup \{\{a,c\},\{c,b\}\})$ is a 2-link. Assume it is also true for $n = 2m-1, 2m$ for some $m > 1$. Now consider $n = 2m+1, 2m+2$. We are going to start with an n-wheel-sunflower with a missing petal, $WS_n \setminus \{\{a,c\},\{c,b\}\}$, and trace the rim edge $\{a,b\}$ without a petal. Thus by applying Proposition 4.1.5(v) we have Equation 4.2.2 from Figure 4.12:

$$\begin{aligned} L(K(WS_n \setminus \{\{a,c\},\{c,b\}\})) &= L(K((WS_n \setminus \{\{a,c\},\{c,b\}\}) \setminus \{a,b\})) \\ &+ L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\})). \end{aligned} \tag{4.2.2}$$

Thus $(WS_n \setminus \{\{a,c\},\{c,b\}\}) \setminus \{a,b\}$ is an n-fan-sunflower, F_n, so we can compute

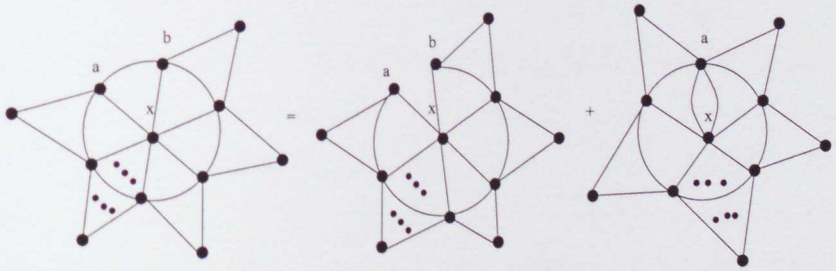

FIGURE 4.12. Computation of $L(K(WS_d \setminus \{\{a,c\},\{c,b\}\}$

$L(K(F_n))$ by Proposition 4.2.8. However $L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\}))$ can be simplified by deleting the parallel pair of edges using Proposition 4.1.7(ii). We thus get an $(n-1)$-wheel-sunflower with a missing spoke, $WS_{(n-1)} \setminus \{a,x\}$. Hence $L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\})) = L(K(WS_{(n-1)} \setminus \{a,x\}))$ And we can compute $L(K(WS_{(n-1)} \setminus \{a,x\}))$ by Proposition 4.2.2. Thus by substituting these back into the Equation 4.2.2 and letting $WS_d \setminus \{\{a,c\},\{c,b\}$ be a d-wheel-sunflower with a missing petal we get Equation 4.2.3:

$$L(K(WS_d \setminus \{\{a,c\},\{c,b\}\})) = L(K(F_n)) + L(K(WS_{(n-1)} \setminus \{a,x\})). \tag{4.2.3}$$

But we know that $L(K(F_d)) = L(K(F_{(d+2)}))$, from Proposition 4.2.8. Hence $L(K(WS_d \setminus \{\{a,c\},\{c,b\}\})) = L(K(WS_{(d-2)} \setminus \{\{a,c\},\{c,b\}\}))$. Thus for $d = 2m+1$ and $d = 2m+2$ we

PROPOSITION 4.2.9. *Let $WS_n \setminus \{\{a,c\},\{c,b\}\}$ be an n-wheel-sunflower with a missing petal $\{\{a,c\},\{c,b\}\}$ and $K(WS_n \setminus \{\{a,c\},\{c,b\}\})$ be its corresponding link. Then $K(WS_n \setminus \{\{a,c\},\{c,b\}\})$ is a 2-link if n is even. Otherwise it is a knot.*

PROOF. We use induction on n. For $n = 2$, then $WS_2 \setminus \{\{a,c\},\{c,b\}\}$ is a 2-wheel with one petal $W_2 \cup \{\{a,c\},\{c,b\}\}$. Hence by Proposition 4.2.6, we know that $K(W_2 \cup \{\{a,c\},\{c,b\}\})$ is a 2-link. Assume it is also true for $n = 2m-1, 2m$ for some $m > 1$. Now consider $n = 2m+1, 2m+2$. We are going to start with an n-wheel-sunflower with a missing petal, $WS_n \setminus \{\{a,c\},\{c,b\}\}$, and trace the rim edge $\{a,b\}$ without a petal. Thus by applying Proposition 4.1.5(v) we have Equation 4.2.2 from Figure 4.12:

$$\begin{aligned} L(K(WS_n \setminus \{\{a,c\},\{c,b\}\})) &= L(K((WS_n \setminus \{\{a,c\},\{c,b\}\}) \setminus \{a,b\})) \\ &+ L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\})). \end{aligned} \tag{4.2.2}$$

Thus $(WS_n \setminus \{\{a,c\},\{c,b\}\}) \setminus \{a,b\}$ is an n-fan-sunflower, F_n, so we can compute

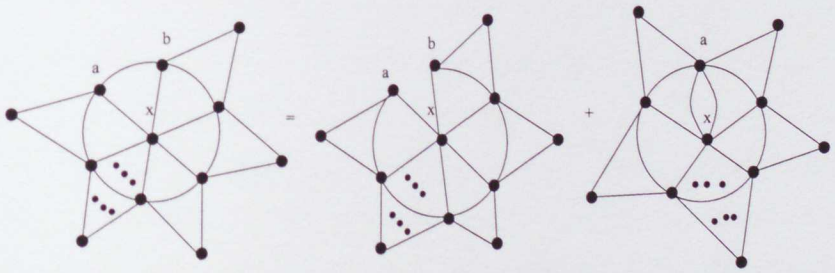

FIGURE 4.12. Computation of $L(K(WS_d \setminus \{\{a,c\},\{c,b\}\}$

$L(K(F_n))$ by Proposition 4.2.8. However $L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\}))$ can be simplified by deleting the parallel pair of edges using Proposition 4.1.7(ii). We thus get an $(n-1)$-wheel-sunflower with a missing spoke, $WS_{(n-1)} \setminus \{a,x\}$. Hence $L(K((WS_n \setminus \{\{a,c\},\{c,b\}\})/\{a,b\})) = L(K(WS_{(n-1)} \setminus \{a,x\}))$ And we can compute $L(K(WS_{(n-1)} \setminus \{a,x\}))$ by Proposition 4.2.2. Thus by substituting these back into the Equation 4.2.2 and letting $WS_d \setminus \{\{a,c\},\{c,b\}$ be a d-wheel-sunflower with a missing petal we get Equation 4.2.3:

$$L(K(WS_d \setminus \{\{a,c\},\{c,b\}\})) = L(K(F_n)) + L(K(WS_{(n-1)} \setminus \{a,x\})). \tag{4.2.3}$$

But we know that $L(K(F_d)) = L(K(F_{(d+2)}))$, from Proposition 4.2.8. Hence $L(K(WS_d \setminus \{\{a,c\},\{c,b\}\})) = L(K(WS_{(d-2)} \setminus \{\{a,c\},\{c,b\}\}))$. Thus for $d = 2m+1$ and $d = 2m+2$ we

have:

$$\begin{aligned} L(K(WS_{(2m+1)} \setminus \{\{a,c\},\{c,b\})) &= L(K(WS_{(2m)} \setminus \{\{a,c\},\{c,b\})), \\ L(K(WS_{(2m+2)} \setminus \{\{a,c\},\{c,b\})) &= L(K(WS_{(2m+1)} \setminus \{\{a,c\},\{c,b\})). \end{aligned}$$

Therefore, the proposition holds for any n. □

Conclusion

This thesis discusses a new class of graphs called wheel-sunflowers. The Tutte polynomials of wheel-sunflowers and their applications have also been considered. We now review the results obtained and give some suggestions for future research.

Chapter two gives the definition of wheel-sunflower graphs, their properties and characterizations of the cycle matroid $M(WS_n)$ of an n-wheel-sunflower. The characterizations are used to compute the Tutte polynomial of $M(WS_n)$ from Tutte polynomials of generalized parallel connections [**3**]. It would be interesting though to find the explicit expression for Tutte polynomials of wheel-sunflowers. Such an expression may be useful to evaluate reliability of *wheel-sunflower networks* in random graph theory.

Chapter three applies the theory of Tutte polynomials to prove T-uniqueness of wheel-sunflowers. We have proved that for $n \geq 4$, WS_n is T-unique. The proof mimicks the proof of T-uniqueness of graphs such as wheels. T-unique graphs and matroids have received much attention [**11, 12, 13**]. It could be more interesting to consider T-uniqueness of wheel-sunflowers using other polynomials such as the chromatic polynomial and the matching polynomial.

Chapter four also applies the theory of Tutte polynomials to classify links associated with wheel-sunflowers and their variations. In this chapter, we have proved that the link $K(WS_n)$ coresponding to an n-wheel-sunflower is a 3-link if n is even. Otherwise $K(WS_n)$ is a 2-link. We have also shown how these links change with removal of certain edges in WS_n. For example, by removing one "outer" triangle from WS_n, we obtain a graph which we have called an n-Fan-sunflower, F_n. The link $K(F_n)$ corresponding to F_n is a 2-link if n is even. Otherwise $K(F_n)$ is a knot. It would be interesting to consider k-sums of wheel-sunflowers and their corresponding links.

Index of Definitions

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Appendix

FIGURE 4.13. Table of Knots by Dale Rolfsen, 1976